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Solutions of the Yang–Baxter equation for isotropic quantum spin chains

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Abstract. We consider solutions of the Yang–Baxter equation such that the logarithmic derivative of the transfer matrix yields a quantum spin Hamiltonian which is isotropic in spin space, i.e. SU(2)-invariant. Four such solutions are known for each value of the spin S. (For $S = \frac{1}{2}$ they degenerate into the same solution, and for $S = 1$ they only give three different solutions.) For $S \leqslant 6$ we show that these are the only solutions which are SU(2)-invariant, except for $S = 3$ when there is a fifth solution.

1. Introduction

The motivation for this paper is the following question. Which nearest neighbour, translation-invariant quantum spin chains which are isotropic in spin space are solvable? By solvable we mean that there is a solution of the Yang–Baxter equation with 'difference variables' which yields the Hamiltonian of the quantum spin chain in the usual way. By isotropic in spin space we mean that the solution has an SU(2) symmetry. Letting S denote the spin of the chain, we completely answer this question for $S \leqslant 6$, i.e. we list all the solvable isotropic chains. The number of solvable Hamiltonians is 1 for $S = \frac{1}{2}$, 3 for $S = 1$, and 4 for $\frac{3}{2} \leqslant S \leqslant 6$ except for $S = 3$ when the number is 5. All of the solutions in this list were previously known with the possible exception of the extra solution that appears at $S = 3$. We should emphasize two points. First, when we say a Hamiltonian is solvable we mean only that there is a solution of the Yang–Baxter equation corresponding to this Hamiltonian. Using this solution to actually solve the model, i.e. to compute the ground-state energy, gaps etc, is quite non-trivial, and we do not address this problem here. Second, spin chains may be solvable in a sense different from that considered here. For example the one-dimensional Hubbard model is solvable [1], but does not correspond to a solution of the Yang–Baxter equation in the form we consider here, even though there is a one-parameter family of operators that commute with the Hamiltonian [2].

We begin with a brief discussion of the Yang–Baxter equation and the correspondence between a solution and a solvable quantum spin Hamiltonian. We then review the four known series of isotropic solutions of the Yang–Baxter equation. For $S = 3$ we present a fifth solution. For $S \leqslant 6$ we show (with some numerical help from the computer) that all the isotropic solutions have been found. Since a solution to the Yang–Baxter equation is essentially determined by the associated quantum spin Hamiltonian, this means that for $S \leqslant 6$ all isotropic solutions of the Yang–Baxter equation are known.

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2. The Yang–Baxter equation

Let $R(\lambda)$ be a one-parameter family of linear operators which act on $\mathbb{C}^k \otimes \mathbb{C}^k$. Consider the three-fold tensor product $\mathbb{C}^k \otimes \mathbb{C}^k \otimes \mathbb{C}^k$. Let $R_{12}(\lambda)$ denote $R(\lambda)$ acting on the first two factors, i.e. $R(\lambda) \otimes 1$ where 1 is the identity operator on $\mathbb{C}^k$. $R_{23}(\lambda)$ denotes $R(\lambda)$ acting on the second and third factors, i.e. $1 \otimes R(\lambda)$. The Yang–Baxter equation [3–5] can be written as follows:

$$R_{12}(\lambda)R_{23}(\lambda+\mu)R_{12}(\mu) = R_{23}(\mu)R_{12}(\lambda+\mu)R_{23}(\lambda). \tag{1}$$

An equivalent form of this equation is

$$\bar{R}_{12}(\lambda)\bar{R}_{13}(\lambda+\mu)\bar{R}_{23}(\mu) = \bar{R}_{23}(\mu)\bar{R}_{13}(\lambda+\mu)\bar{R}_{12}(\lambda). \tag{2}$$

These two equations are equivalent in the sense that $R(\lambda)$ satisfies (1) if and only if $\bar{R}(\lambda) = ER(\lambda)$ satisfies (2) where E is the operator that exchanges the two factors in $\mathbb{C}^k \otimes \mathbb{C}^k$.

To obtain a quantum spin chain Hamiltonian from an $R(\lambda)$ we need the following condition on $R(\lambda)$. We use the terminology of [6].

Definition. A solution $R(\lambda)$ is said to be *regular* if $R(0) = 1$.

Note that if one uses the other form of the Yang–Baxter equation (equation (2)) then the solution $\bar{R}(\lambda)$ is regular if and only if $\bar{R}(0) = E$. (This is the convention followed in [6].) If $R(\lambda)$ is regular, then the logarithmic derivative of the transfer matrix which is constructed from $R(\lambda)$ in the usual way is $\sum_i H_{i,i+1}$ where $H = \mathrm{d}R/\mathrm{d}\lambda(0)$ and $H_{i,i+1}$ denotes H acting on sites i and $i+1$ [7]. This logarithmic derivative is the solvable quantum spin chain that goes with the solution $R(\lambda)$.

A couple of trivial comments are in order. If $R(\lambda)$ is a regular solution of the Yang–Baxter equation then for any scalar function $f(\lambda)$ with $f(0) = 1$, $f(\lambda)R(\lambda)$ is again a regular solution of the Yang–Baxter equation. The quantum spin Hamiltonian associated with $f(\lambda)R(\lambda)$ equals the Hamiltonian associated with $R(\lambda)$ plus a constant. If $R(\lambda)$ is a regular solution of the Yang–Baxter equation then so is $R(\alpha\lambda)$ where α is any constant. The Hamiltonian associated with this solution is just α times the Hamiltonian associated with the original solution. Thus if H is solvable in the sense of corresponding to a solution of the Yang–Baxter equation, then so is $\alpha H + c$ for any constants α and c. When we count how many solvable Hamiltonians there are, we will consider all the Hamiltonians $\alpha H + c$ as the same Hamiltonian.

We are interested in quantum spin chains which are isotropic in spin space. This means that there is an irreducible representation of SU(2) on $\mathbb{C}^k$. Thus there are spin operators $\boldsymbol{S} = (S^x, S^y, S^z)$ acting on $\mathbb{C}^k$ which obey the usual commutation relations, and $\boldsymbol{S} \cdot \boldsymbol{S} = S(S+1)$ where $2S + 1 = k$. We let $\boldsymbol{S}_i$ denote $\boldsymbol{S}$ acting on the ith factor of the tensor product.

Definition. If H acts on $\mathbb{C}^k \otimes \mathbb{C}^k$, we say that H is *isotropic* if $[H_{12}, \boldsymbol{S}_1 + \boldsymbol{S}_2] = 0$.

All isotropic operators on $\mathbb{C}^k \otimes \mathbb{C}^k$ can be written as polynomials in $\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$. Powers of $\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$ greater than $2S$ can be written as linear combinations of powers less than or equal to $2S$. Thus an isotropic H can be written as $H_{12} = \sum_{l=0}^{2S} h_l (\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)^l$

where the h_l are scalars. We are only interested in self-adjoint Hamiltonians, so we require the h_l to be real. For $k = 0, 1, 2, \dots, 2S$, let $P^{(k)}$ denote the operator on two sites that is the projection onto states with total spin k on the two sites. Then a self-adjoint, isotropic H can also be written as $H = \sum_{k=0}^{2S} c_k P^{(k)}$ for some real constants c_k.

3. Particular isotropic solutions of the Yang–Baxter equation

The simplest isotropic solution of the Yang–Baxter equation is just the following:

$$R(\lambda) = 1 + \lambda E\,. \tag{3}$$

This solution appears in McGuire [3] and later in Yang [4]. The quantum spin-Hamiltonian that comes from this solution is given by

$$H = \sum_i E_{i,i+1} \tag{4}$$

where $E_{i,i+1}$ interchanges the spins at sites i and $i+1$. This Hamiltonian was studied by Uimin [8], Lai [9] and Sutherland [10].

Zamolodchikov and Zamolodchikov [11] found a factorizable S-matrix which has $O(N)$ symmetry. Their S-matrix may be rewritten as an isotropic, i.e. SU(2)-invariant, solution of the Yang–Baxter equation

$$R(\lambda) = 1+\lambda\{1+[S+\tfrac{1}{2}-(-1)^{2S}]E-(-1)^{2S}(2S+1)P^{(0)}\}+\lambda^2[S+\tfrac{1}{2}-(-1)^{2S}]E\,. \tag{5}$$

(Recall that $P^{(0)}_{i,i+1}$ denotes the projection onto states whose restriction to sites i and $i+1$ has total spin zero.) The quantum spin Hamiltonian corresponding to this solution of the Yang–Baxter equation is found by computing $R'(0)$. It is

$$H = \sum_i \{[S+\tfrac{1}{2}-(-1)^{2S}]E_{i,i+1} - (-1)^{2S}(2S+1)P^{(0)}_{i,i+1}\}\,. \tag{6}$$

We have dropped the trivial constant term.

Since the equivalence of the above solution and the factorizable S-matrix of [11] may not be transparent, we briefly show how one may verify directly that (5) is a solution of the Yang–Baxter equation. In the usual convention of S^z eigenstates we let $|j,k\rangle$ denote the state on two sites with $S^z = j$ on the left site and $S^z = k$ on the right site. (Here $j,k,l = S, S-1, S-2, \dots, 2, 1, 0$ if $2S$ is even and $j,k,l = S, S-1, S-2, \dots, \frac{3}{2}, \frac{1}{2}$ if $2S$ is odd.) The only non-zero matrix elements of $P^{(0)}$ are given by

$$\langle k, -k \mid P^{(0)} \mid l, -l\rangle = (-1)^{l-k}/(2S+1)\,.$$

Using this equation and letting $U = (2S+1)P^{(0)}$, one can verify the following relations:

$$
\begin{aligned}
&U^2 = (2S+1)U \\
&EU = UE = (-1)^{2S}U \\
&U_{12}E_{23}U_{12} = U_{12} \\
&E_{12}U_{23}U_{12} = (-1)^{2S}E_{23}U_{12} \\
&U_{12}U_{23}E_{12} = (-1)^{2S}U_{12}E_{23} \\
&U_{12}U_{23}U_{12} = U_{12}\,.
\end{aligned}
\tag{7}
$$

These relations and a lot of tedious algebra verify that (5) is a solution of the Yang–Baxter equation.

Another isotropic solution of the Yang–Baxter equation was found by Kulish *et al* [12]. Multiplying their solution by a scalar function, it can be written as

$$
R(\lambda) = \sum_{k=0}^{2S}\left[\prod_{j=1}^{k}(j-\lambda)\prod_{j=k+1}^{2S}(\lambda+j)\right]P^{(k)}\,. \tag{8}
$$

Up to constants that do not matter, the solvable quantum spin-Hamiltonian corresponding to this solution is

$$
H = \sum_i \sum_{k=0}^{2S}\left(\sum_{j=1}^{k}\frac{1}{j}\right)P^{(k)}_{i,i+1}\,. \tag{9}
$$

This Hamiltonian was studied by Babudjian [13] and Takhatajan [14].

Schultz [15] and Perk and Schultz [16] found several families of solvable models. One of these models yields the following isotropic quantum spin Hamiltonian:

$$
H = \sum_i P^{(0)}_{i,i+1}\,. \tag{10}
$$

(The solvablility of this model for $S=1$ goes back to Stroganov [17].) This Hamiltonian was studied in [18] and later in [19–21]. As observed in [18] and later in [19, 20], one way to understand the solvability of this model is that it yields a representation of the Temperley–Lieb algebra [22]. The relations defining this algebra are

$$
\begin{aligned}
&U^2_{i,i+1} = \sqrt{q}\,U_{i,i+1} \\
&U_{i,i+1}\,U_{i+1,i+2}\,U_{i,i+1} = U_{i,i+1} \\
&[U_{i,i+1}, U_{j,j+1}] = 0 \qquad \text{if} \quad |i-j| > 1\,.
\end{aligned}
$$

If we let $U_{i,i+1} = (2S+1)P^{(0)}_{i,i+1}$, then the $U_{i,i+1}$ are a representation of this algebra.

Any representation of the Temperley–Lieb algebra yields a solution of the Yang–Baxter equation in the following way [23]. If $U_{i,i+1}$ is an operator which acts on sites i and $i+1$ and satisfies the above relations, then

$$
R(\lambda) = 1 + f(\lambda)U
$$

is a solution of the Yang–Baxter equation provided $f(\lambda)$ satisfies the equation

$$f(\lambda) + f(\mu) + \sqrt{q} f(\lambda) f(\mu) + f(\lambda) f(\mu) f(\lambda+\mu) - f(\lambda+\mu) = 0\,.$$

This equation always has a solution, although the nature of the solution depends on whether q is greater than 4, less than 4 or equal to 4. In the case at hand, $q = (2S+1)^2$ and for $S > \frac{1}{2}$, $f(\lambda) = (a - a\mathrm{e}^{\lambda})/(\mathrm{e}^{\lambda} - a^2)$ where a is a solution of $a + 1/a = 2S+1$. Thus the following is a solution of the Yang–Baxter equation for $S \geqslant 1$:

$$R(\lambda) = 1 + \frac{a - a\mathrm{e}^{\lambda}}{\mathrm{e}^{\lambda} - a^2}(2S+1)P^{(0)}\,. \tag{11}$$

For $S = \frac{1}{2}$ there is essentially only one isotropic Hamiltonian, and so all four series of solutions yield this Hamiltonian. When $S = 1$ the solutions (5) and (8) are the same, so there are three solutions. For $S \geqslant \frac{3}{2}$ the four solutions are different.

When $S = 3$, there is a fifth solution in addition to the four solutions described above. It is

$$\begin{aligned} R(\lambda) = {} & -9(\lambda+1)(\lambda-\tfrac{2}{3})(\lambda+\tfrac{1}{6})P^{(0)} + 9(\lambda-1)(\lambda-\tfrac{2}{3})(\lambda+\tfrac{1}{6})(P^{(1)}+P^{(5)}) \\ & - 9(\lambda-1)(\lambda-\tfrac{2}{3})(\lambda-\tfrac{1}{6})(P^{(2)}+P^{(4)}+P^{(6)}) \\ & + 9(\lambda-1)(\lambda+\tfrac{2}{3})(\lambda-\tfrac{1}{6})P^{(3)}\,. \end{aligned} \tag{12}$$

Checking that this is indeed a solution is simply a matter of computation which is best done with the help of a computer. Up to an overall constant, the quantum spin Hamiltonian that goes with this solution is

$$H = \sum_i \left(11P^{(0)}_{i,i+1} + 7P^{(1)}_{i,i+1} - 17P^{(2)}_{i,i+1} - 11P^{(3)}_{i,i+1} - 17P^{(4)}_{i,i+1} + 7P^{(5)}_{i,i+1} - 17P^{(6)}_{i,i+1}\right). \tag{13}$$

Batchelor and Kuniba have pointed out that this R-matrix appears to be the rational limit of the R-matrix corresponding to the Lie algebra G_2 which was constructed in [24].

4. Classification of isotropic solutions

We now turn to the question of whether or not we have listed all the solvable isotropic, i.e. SU(2)-invariant, quantum spin Hamiltonians in the previous section. The Yang–Baxter equation puts severe constraints on H. We will use a constraint derived by Reshetikin. We begin with a quick review of this condition, following [6]. The idea is to expand the Yang–Baxter equation in powers of λ and μ. Let $R(\lambda)$ be a regular solution of the Yang–Baxter equation that is analytic in a neighbourhood of $\lambda = 0$. We can write its power series in the form

$$R(\lambda) = 1 + \lambda H + \sum_{n=2}^{\infty} \lambda^n R^{(n)}\,.$$

The $\lambda\mu$ equation implies

$$R^{(2)} = \tfrac{1}{2}H^2 + c$$

for some constant c. Using this relation the $\lambda\mu^2$ equation can be written as

$$R^{(3)}_{12} - R^{(3)}_{23} + \tfrac{1}{6}[H_{12} + H_{23}, [H_{12}, H_{23}]] - \tfrac{1}{6}H^3_{12} + \tfrac{1}{6}H^3_{23} = 0\,.$$

Thus there is an operator X which acts only on two sites such that

$$[H_{12} + H_{23}, [H_{12}, H_{23}]] = X_{23} - X_{12}\,. \tag{14}$$

This is Reshetikin's necessary condition for the existence of a solution to the Yang–Baxter equation corresponding to the Hamiltonian H.

Let D be an operator that acts on three sites, and let D_{123} denote this operator acting on sites 1, 2 and 3. We want to study the equation

$$D_{123} = X_{23} - X_{12}\,. \tag{15}$$

On dimensional grounds alone we see that for most D this equation will not have a solution. When there is a solution, we want to know to what extent this equation determines X. Let tr_i denote the trace over the state space at site i, normalized so that $\mathrm{tr}_i 1 = 1$. Let tr_{ij} denote the trace over the state spaces at site i and j, normalized so that $\mathrm{tr}_{ij} 1 = 1$. (Note that $\mathrm{tr}_{ij} A = \mathrm{tr}_i(\mathrm{tr}_j A)$ for any operator A.)

Lemma 1. If X is an operator on two sites which satisfies (15), then

$$X_{12} = -\mathrm{tr}_3 D_{123} - \mathrm{tr}_{34} D_{234} + cI \tag{16}$$

where c is an undetermined constant and I is the identity operator. In particular, if $X_{23} - X_{12} = 0$ and $\mathrm{tr}_{12} X_{12} = 0$, then $X = 0$.

Proof. If X is a solution of (15) then so is $X + cI$ for any constant c. So we can assume $\mathrm{tr}_{12} X_{12} = 0$. Applying tr_{23} to (15) we have

$$\mathrm{tr}_2 X_{12} = -\mathrm{tr}_{23} D_{123}\,.$$

Shifting this equation we have

$$\mathrm{tr}_3 X_{23} = -\mathrm{tr}_{34} D_{234}\,.$$

If we now apply tr_3 to (15) we obtain (16). □

The lemma says we can replace $X_{23} - X_{12}$ in (14) by an expression that only involves H. So (14) is a homogeneous equation of degree three for H. One can now play the following game. Consider all the H which can be written as $H = \sum_{\alpha=1}^{n} c_\alpha H_\alpha$ where the H_α are some fixed set of linearly independent operators on two sites and the c_α are real numbers. Then the matrix equation (14) becomes a system of homogenous equations of degree three in the parameters c_α. One can then try and find all solutions of this system of equations. Reshetikin's condition (14) is a necessary one for a solution to the Yang–Baxter equation corresponding to the Hamiltonian to exist, but it is not known if it is sufficient. However, this is not a problem if for each solution of (14) one can find (or already has) a solution of the Yang–Baxter equation.

Unless n is fairly small, one typically obtains a huge number of equations in a large number of variables. However, for isotropic Hamiltonians something special happens which makes these equations easy to solve. All the isotropic Hamiltonians for spin S are of the form

$$H = \sum_{\alpha=0}^{2S} c_\alpha P^{(\alpha)} \tag{17}$$

where $P^{(\alpha)}$ is the projection onto states whose total spin on the two sites is α. Since (14) is unchanged if we shift H by a multiple of the identity, we may take $c_{2S} = 0$ without loss of generality.

Let $|j,k,l\rangle$ denote the state with $S^z = j,k,l$ on sites 1, 2, 3. We take H of the form (17) and consider the matrix element of (14) of the form $\langle S,j,k|\,|j+k-S,S,S\rangle$ with $k < S$. Clearly $\langle S,j,k \mid X_{23} - X_{12} \mid j+k-S,S,S\rangle = 0$. So we have

$$\langle S,j,k \mid [H_{12} + H_{23}, [H_{12}, H_{23}]] \mid j+k-S,S,S\rangle = 0\,.$$

All of the operators involved in this equation conserve the total S^z. So the three sites always have a total S^z of $S+j+k$. Thus the smallest that the total S^z can be on any two sites is $j+k$. This implies that no c_α with $\alpha < j+k$ appears in this equation. Thus for $l = 2S-2, 2S-3, \ldots, 2, 1, 0$, we can find equations that contain only $c_{2S-1}, c_{2S-2}, \ldots c_l$. By homogeneity we can assume c_{2S-1} is either 1 or 0. Then we can use the equations to solve for $c_{2S-2}, c_{2S-3}, \ldots$ successively. *A priori* this requires solving a cubic equation to find c_l, but it turns out that the equation that contains only $c_{2S-1}, c_{2S-2}, \ldots c_l$ is only quadratic in c_l. The number of equations is greater than the number of unknowns, so after solving for the c_α we check that they satisfy the remaining equations. Most of the time they do not. We have carried out this program on a computer for S up to 6. Of course every solution of the Yang–Baxter equation discussed in the previous section gives a solution of the equations for the c_αs. The final result of the computer calculations is that these are the only solutions.

Quasi-theorem 2. For $S \leqslant 6$ the only isotropic, i.e. SU(2)-invariant, nearest-neighbour, self-adjoint quantum spin Hamiltonians which correspond to a regular solution of the Yang–Baxter equation are those discussed above. Specifically, the Hamiltonians are

(i) $S = \frac{1}{2}$: one Hamiltonian—(4)
(ii) $S = 1$: three Hamiltonians—(4), (6), (10)
(iii) $\frac{3}{2} \leqslant S \leqslant 6$, $S \neq 3$: four Hamiltonians—(4), (6), (9), (10)
(iv) $S = 3$: five Hamiltonians—(4), (6), (9), (10), (13).

We have described this result as a 'quasi-theorem' because of the use of the computer. Because of rounding errors the equations are not satisfied exactly but only to some high degree of accuracy. Thus it is conceivable that there is a solution that actually satisfies the equations, but the computer mistakenly concludes that it does not because of the rounding errors. We have taken care to ensure that this does not happen, but we cannot claim to have truly proved the above result.

As we noted before, if $R(\lambda)$ is a regular solution of the Yang–Baxter equation, then for any scalar function $f(\lambda)$ with $f(0) = 1$, $f(\lambda)R(\lambda)$ is also a regular solution. The quantum spin Hamiltonian associated with a solution $R(\lambda)$ determines $R(\lambda)$ up to this freedom to multiply by a scalar function. Although this fact is probably well known, we have been unable to find it in the literature, so we include a precise statement and proof.

Theorem 3. Let $R(\lambda)$ and $Q(\lambda)$ be regular solutions of the Yang-Baxter equation which are analytic in a neighbourhood of $\lambda = 0$. If $R'(0) = Q'(0)$ then there is a scalar function $f(\lambda)$ which is analytic in a neighbourhood of $\lambda = 0$ such that

$$R(\lambda) = f(\lambda)Q(\lambda)$$

in a neighbourhood of $\lambda = 0$.

Proof. Let

$$\bar{R}(\lambda) = R(\lambda)/\mathrm{tr}\, R(\lambda)$$
$$\bar{Q}(\lambda) = Q(\lambda)/\mathrm{tr}\, Q(\lambda)\,.$$

(Since $\mathrm{tr}\, R(0) = \mathrm{tr}\, Q(0) \neq 0$, these are analytic in a neighbourhood of 0.) Clearly $\mathrm{tr}\bar{R}(\lambda) = \mathrm{tr}\bar{Q}(\lambda) = 1$, so if we write

$$\bar{R}(\lambda) = \sum_{n=0}^{\infty} \lambda^n R^{(n)}$$

$$\bar{Q}(\lambda) = \sum_{n=0}^{\infty} \lambda^n Q^{(n)}$$

then we have $\mathrm{tr}\bar{R}^{(n)} = \mathrm{tr}\bar{Q}^{(n)} = 0$ for $n \geqslant 1$.

We now proceed by induction. Assume $\bar{Q}^{(k)} = \bar{R}^{(k)}$ for $k < n$. We expand the Yang–Baxter equation in a power series in λ and μ. Consider the $\lambda\mu^{n-1}$ terms. They will only involve $R^{(k)}$ with $k \leqslant n$. Furthermore, $R^{(n)}$ only appears in the form $R_{23}^{(n)} - R_{12}^{(n)}$. Thus by the inductive assumption

$$(\bar{R}_{23}^{(n)} - \bar{Q}_{23}^{(n)}) - (\bar{R}_{12}^{(n)} - \bar{Q}_{12}^{(n)}) = 0\,.$$

Since $\mathrm{tr}(\bar{R}^{(n)} - \bar{Q}^{(n)}) = 0$, lemma 1 implies $\bar{R}^{(n)} - \bar{Q}^{(n)} = 0$. Thus $\bar{R}(\lambda) = \bar{Q}(\lambda)$, which proves the theorem. □

Combining quasi-theorem 2 with the above theorem we have the following.

Quasi-theorem 4. For $S \leqslant 6$ the only isotropic, i.e. SU(2)-invariant, regular, self-adjoint solutions of the Yang-Baxter equation are of the form $f(\lambda)R(\lambda)$ where $f(\lambda)$ is a scalar function and $R(\lambda)$ is one of the solutions discussed above. Specifically, the only possibilities for $R(\lambda)$ are

(i) $S = \frac{1}{2}$: equation (3)
(ii) $S = 1$: equations (3), (5), (11)
(iii) $\frac{3}{2} \leqslant S \leqslant 6$, $S \neq 3$: equations (3), (5), (8), (11)
(iv) $S = 3$: equations (3), (5), (8), (11), (12).

The proof of theorem 3 shows that given the quantum spin Hamiltonian one can explicitly compute the solution $R(\lambda)$ of the Yang–Baxter equation order by order in λ. One can program a computer to carry out this calculation. If the solution is rational (and so equivalent to a polynomial solution), then the computer can compute the entire solution.

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